results about them. The function field $K(X)$ is a subfield of $K(X_{/Y})$; one of the important questions is: when are $K(X)$ and $K(X_{/Y})$ equal, in other words, when can all formal-rational functions on $X_{/Y}$ be extended to X? Many of the theorems state suitable conditions on X and Y under which $K(X) = K(X_{/Y})$.

The other chapters in the second part deal with selected topics from formal geometry such as Lefschetz theory, formal functions on homogeneous spaces and quasi-lines on projective varieties.

The stated prerequisite is [**1**] or a similar text, but otherwise the book is self-contained. It is suitable for advanced graduate students or researchers in algebraic geometry or complex geometry.

Reference

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INVARIANT MANIFOLDS FOR PHYSICAL AND CHEMICAL KINETICS
(Lecture Notes in Physics 660)

By A. N. Gorban and I. V. Karlin: 495 pp., £77,

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1. *Introduction*

Statistical mechanics is the branch of theoretical physics which endeavours to explain the macroscopic behaviour of natural systems in terms of the microscopic interactions between their fundamentals constituents. It comes in two major flavors: equilibrium and non-equilibrium. At equilibrium, only time-asymptotic states are relevant to the macroscopic description, and consequently the time-evolution of the system plays no role. As a result, many elegant and general results can be established, which place equilibrium statistical mechanics on a very solid foundational and operational basis. Non-equilibrium statistical mechanics, especially for systems *far-off* equilibrium, tells a fairly different story. For such systems, the dynamics does play a vital role, and the task of establishing general principles is consequently a much harder and way less accomplished one. One of the few powerful and general ideas which help to unravel the complexities of non-equilibrium systems is the notion of a separation between 'slow' and 'fast' degrees of freedom. Loosely speaking, the former describe the large-scale, macroscopic dynamics, whereas the latter are associated with microscopic fluctuations around the slow modes. Of course, the 'slow/fast' dichotomy is a rather fuzzy notion, and one which necessitates quantitative specification in order to acquire genuine scientific status. One such quantitative underpinning is provided by the theory of slow invariant manifolds, that is, submanifolds of the system phase-space which act as attractors of the long-time dynamics of the system. Providing criteria to identify, construct and classify the slow invariant manifolds of complex systems, thereby offering a quantitative tool for the investigation of non-equilibrium statistical systems, is a task of the utmost importance in modern non-equilibrium statistical mechanics. This task makes up the central core of the very remarkable monograph by Gorban and Karlin.

2. *Finding the manifold: the invariance condition*

Let us consider a dynamical system described by a state variable x belonging to a generic N-dimensional phase-space U. Let the evolution of the system be described by the following first-order initial value problem:

$$\frac{dx}{dt} = J(x); \tag{1}$$

$$x(0) = x_0. \tag{2}$$

The vector field $J(x)$ generates the motion in a phase space U. For many (dissipative) systems, this motion is such that after a rapid transient, and regardless of the initial position x_0, the trajectory $x(t)$ 'lands' into a lower-dimensional manifold $\Omega \in U$, never to quit it again for the rest of its lifetime.

The latter 'land-and-stay' condition is precisely what defines the invariance of the manifold Ω. This very qualitative description alone already highlights the fundamental importance of low-dimensional invariant manifolds: they permit us to describe the time-evolution of complex systems in terms of many fewer variables than those needed to characterize the full phase-space U. In other words, the manifold is where the system spends most of its time, and is consequently where relevant things happen. It is a general and profound trait of complex dynamical systems that relevant phenomena take place only in a very minute portion of the complete phase-space available to them. Spotting that tiny relevant portion is therefore a task of the utmost importance.

The next question is: how do we identify the invariant manifolds associated with the dynamical system defined by the equation (1)? To this purpose, let us, for any point x of Ω, split the phase-space onto the Ω-tangent space, T_Ω, and its complement. Any velocity vector J is hence split into a parallel component, $J_\parallel$, running along T_Ω, and $J_\perp$, spanning the complementary subspace. This splitting is formally performed via a projection operator P, such that:

$$J_\parallel = PJ, \qquad J_\perp = QJ,$$

where $Q = 1 - P$ is the co-projector. By acting with P, Q on both sides of 1, we obtain a reduced vector field tangent to manifold $PJ(x)$ and a vector field that is transversal to manifold $QJ(x)$. The 'land-and-stay' condition implies that should a point x belong to a manifold at a certain time $t_0 > 0$, it will stay thus at any subsequent time $t > t_0$. This requires its time derivative to be identically zero at all times; that is:

$$\Delta(x) \equiv QJ(x) = 0, \tag{3}$$

where $\Delta(x)$ is the so-called *defect of invariance*, a self-explanatory name indeed. Thus, the condition $\Delta(x) = 0$ *defines* the invariant manifold. Equation (3) provides a nice mathematical formulation of the invariance property, and yet a purely formal one, until an *operational* procedure is supplied to construct the projector P. On the way towards this constructive task, it is convenient to formulate the problem in differential form. To this purpose, let us parameterize the manifold Ω through a set of coordinates $y \in W$, such that each point x on the manifold is an image of some $y \in W$. The coordinate mapping F is assumed to be an immersion, meaning by this that its differential DF is an isomorphism onto T_Ω. Application of the chain rule to $dx_\parallel/dt = PJ(x)$ provides the induced dynamics in W; that is:

$$\frac{dy}{dt} = (DF)^{-1}\, PJ(F(y)), \tag{4}$$

where DF denotes the Jacobian of J at point x. Its inverse $(DF)^{-1}$ is then defined on the tangent space $T_{x=F(y)}$, the fiber bundle associated with the manifold Ω.

The standard procedure for identifying lower-dimensional invariant manifolds is to perform a spectral analysis of this inverse Jacobian operator (simple eigenvectors corresponding to one-dimensional manifolds).

This is where the fascinating and remarkable story of this book takes off.

Indeed, besides spectral analysis, the book discloses a whole array of powerful *non-perturbative* techniques to solve the invariance equation, often with *no need* to perform any spectral analysis at all. Besides its conceptual appeal, this is far from being a purely academic achievement, since it is well known that computing eigenvalues/vectors in high-dimensional spaces is a very intensive task, which holds back many important applications in modern science and engineering. The book presents three general classes of methods:

(i) iteration methods,
(ii) relaxation methods, and
(iii) natural projector methods,

upon which we now briefly comment.

2.1. *Iteration methods*

Iteration methods tackle the solution of the invariance equation $QJ(x)=0$ by Newton iterations. Both the vector field $J(F(y))$ and the co-projector $Q = 1 - P(F(y))$ depend on the unknown map F, so that one is faced with a non-linear functional problem. At each iteration, the first-order (linear in F) approximation for $J(F(y))$ is used, whereas P is taken at zero-th order (constant). This *incomplete linearization* leads to the slowest invariant manifold, namely the asymptotically stable fixed-point of the film extension. This is a highly non-trivial generalization of the standard notion of linear manifolds associated with leading non-zero eigenvalues of the Jacobian. As is always the case with Newton's method, convergence hinges on the smoothness and regularity of the inverse Jacobian, but practice shows that very often just a few iterations suffice to provide a fairly acceptable approximation.

2.2. *Relaxation methods*

Relaxation methods work in the spirit of 'false-transient' or 'fictitious-time' dynamics. These methods are based on the step-wise solution of the differential equation $dF(y)/dt = \Delta(y)$, the so-called *film-extension* of the dynamics. These methods are very well suited to numerical implementations. In fact, since only the steady-state condition $\Delta = 0$ is of interest, there is no need for them to converge along a realistic physical trajectory. As a result, fictitious dynamics can be designed and efficiently simulated on a computer, typically by marching towards their attractor in much larger steps than those allowed along the physical trajectory, on account of stability constraints. This type of procedure has become quite popular in modern computational statistical mechanics, two notable examples being the Car–Parrinello method for quantum molecular systems, and the lattice Boltzmann method for classical fluid flows.

Of course, here too, things can go wrong if one gets too eager and makes the fictitious dynamics depart too drastically from the physical dynamics. A powerful remedy against these problems, and one for which the authors deserve wide credit, is the discovery, development and application of system-tailored entropies which secure the stability of the fictitious dynamics under quite general conditions. Among others, this has produced the entropic lattice Bolzmann method, possibly the most powerful family of lattice Boltzmann methods to date.

2.3. *Natural projection methods*

Finally, the natural projector method is based on the idea of projecting the dynamics, not on the manifold itself, but rather in terms of a pre-assigned subset of coarse-grained *macroscopic* variables $X = P_n x$, where P_n is the natural projector. Typical examples are the density, velocity and temperature of a fluid flow, as obtained by integration of the Boltzmann distribution function $f(r, v, t)$ over the suitable velocity degrees of freedom. The choice of the subset of natural observables is usually dictated by physical intuition in the first place, mathematical back-up — for all its importance — coming only in a second stage. Typical examples in point are Ehrenfest's coarse-graining, and the Hilbert method for solving the Boltzmann equation. The success of this approach depends crucially on a 'good' choice of the coarse-grained variables, where 'good' means maximal insensitivity to the back-reaction of the decimated degrees of freedom (the 'fast ones'). This insensitivity is a direct measure of the *robustness/persistence* of the invariant manifold, and plays a paramount role in all practical applications of the method.

Each of three methods discussed above opens up countless fascinating connections with many hot topics of modern statistical mechanics and applied mathematics, such as the renormalization group versus Taylor (or Padé) series expansions for the solution of the invariance equation, entropy minimization methods, and many others. It is obviously impossible to to justice to so much beautiful (and thorough) material in just a few pages; thus we shall move on to the second part of the book.

3. *What to do with the manifold: ideas in action*

The second part of the book is devoted to a very practical and crucial question: once the manifold has been identified, what do we do with it? Again, the book provides quite a neat answer: develop models of open systems living in the vicinity of the manifold. The reader is then exposed to a wide series of modern techniques and algorithms to estimate and *improve* the accuracy of the manifold representation towards the true solution $x(t)$. Among others, this takes her/him into the forefront of some of the hottest areas of modern applied mathematics, such as computer-aided multiscale analysis of complex systems and the equation-free approach to coarse-graining. The great strength and unique feature of the book is the amazing wealth of practical and concrete examples which truly show 'ideas in action'! Systems as diverse as hydrodynamic turbulence, chemical reactions, polymer flows, and even quantum systems in Wigner's representation, are all embraced by by the powerful spectrum of constructive methods presented in the book.

4. *Some (friendly) criticism*

As noted above, one the most impressive features of this work is its combination of conceptual depth and practical breadth. Just because of this, it may come as a slight disappointment that the authors did not try to address quantum many-body problems and the whole world of (bio)-chemical applications behind them. For instance, can we use the constructive methods described in this book to advance our current capabilities to compute the ground state of large molecules, and possibly their excited states as well? Can we predict transition rates between metastable states in rugged high-dimensional energy-landscapes such as those associated with large bio-molecules? Despite their quantum nature, all of these problems can be cast in the general form of the equation (1), so that an affirmative answer appears to be plausible. However, rather than a criticism, this is in fact a suggestion for a new book . . . .

5. *Conclusions*

The authors are highly regarded experts in the field, and their remarkable monograph makes a very valuable and timely contribution to modern non-equilibrium statistical mechanics at large. It will be a very useful and enjoyable addition to the library of graduate students and professionals in the very many disciplines — such as applied math, physics, chemistry and biology — that share non-equilibrium statistical mechanics as a common conceptual background for modeling complex system dynamics at large. In this regard, there is no question that the authors have truly honored their own definition of the subject as 'model engineering'. Lucky is the country where engineers can be trained at such a high level!

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LAWS OF SMALL NUMBERS: EXTREMES AND RARE EVENTS
(Second, revised and extended edition)

By Michael Falk, Jürg Hüsler and Rolf-Dieter Reiss: 376 pp., 38.00 euro (CHF 68.00), ISBN 3-7643-2416-3 (Birkhäuser, Basel, 2004).

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This book supplies a thorough and compact account of the more mathematical aspects of the theory of extreme values and rare events. It is a revised and greatly extended version (from 282 pages to 376 pages) of a book first published in 1994. That first edition was based on lectures given at a 1991 conference on 'Laws of